$\overline{X}$ = vector of random inputs about steady state
$\overline{Y}$ = vector of random outputs about steady state
y_i = random output
$\underline{y}$ = average random output
$\overline{Y}^N$ = Nth steady state output vector

Greek Letters

$\overline{\beta}$ = vector of linear constraints
$\overline{\overline{\phi}}$ = correlation matrix
ϕ_{xx} = autocorrelation function
ϕ_{xy} = cross-correlation function
θ_{xx} = spectral density
θ_{xy} = cross-spectral density
$\overline{\overline{\theta}}$ = spectral density matrix
θ = nominal reaction holding time
ρ_f = fluid density

Subscript

$*$ = vector of lower bounds of variable

Superscript

$*$ = vector of upper bounds

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Gas Dynamic Processes Involving Suspended Solids

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Basic gas dynamic equations involving suspended solid particles were formulated. Considerations include momentum and heat transfer between the gaseous and solid phases. Significance of these contributions was illustrated with the case of expansion through a de Laval nozzle. Duct flow and normal shock problems were also discussed. The extent of earlier methods of approximation was pointed out.

Many processes and devices involve gas-solid suspension. A few of these are pneumatic conveying, H-iron process (direct reduction of iron ore), nuclear reactor with gas-solid fuel feeding and/or cooling, nuclear propulsion scheme where ablation of reactor is deliberately allowed for the sake of high performance, and rockets having part of the combustion product in the solid phase. A fundamental study on steady turbulent motion of gas-solid suspension was reported earlier (*1*). Where high speed is involved, such as in some of the above examples, gas dynamic aspects of gas-solid suspensions become significant. The general aspects of motion involve acceleration, friction, heat transfer, and flow discontinuity. This study deals with one-dimensional motion for the sake of simplicity and develops some physical understanding of the contribution of interaction between the gas and the solid particles. The effects of turbulence will be accounted for with generalized parameters.

The basic equations of this study will be applicable to the general problem of one dimensional steady motion involving variation in flow area, an insulated wall or a wall with arbitrary distribution of temperature, friction, and motion in supersonic or subsonic range. The latter is particularly interesting in the present case because of the dispersion and absorption of sound by the solid particles. Therefore the speed of sound, usually a thermodynamic property, depends on the transport of momentum and energy between the two phases in the present case. Since the gas dynamic nature of the gas-solid suspension is most easily seen in nozzle flow process, a nozzle is taken as the major example; this is also consistent with its significance in applications. Flow of a gas-solid suspension through a nozzle has been studied by many, with various methods of approximation (*2*, *3*, *4*). The extent of these approximations will be considered here.

Because of the inertia of solid particles a gas-solid suspension demonstrates an interesting nature of relaxation. The case of the passage of a gas-solid suspension through a shock process is presented in this paper.

All numerical examples are based on a mixture involving 0.3 lb. of magnesia per pound of air, although the methods are applicable to mixtures of any composition.

BASIC EQUATIONS AND SOLUTIONS

To formulate the basic equations the following assumptions are made:

1. There is steady one-dimensional motion, and the effect of turbulence enters only in characteristic parameters.
2. The solid particles are uniformly distributed over each cross section, although it is understood that they are suspended by turbulence and interactions exist between components.
3. The solid particles are uniform in diameter and physical properties. Variations again can be accounted for with characteristic parameters in the following.
4. The drag on the particles is mainly due to differences between the mean velocities of particles and stream. It is expected that the minimum size of solid particles consists of millions of molecules each (even in the submicron range). Hence the velocity of each solid particle due to its thermal state is extremely low. Slip flow, if it occurs, again can be accounted for by an appropriate characteristic parameter.
5. The heat transfer between the gas and the solid is basically due to their mean temperature difference. Effect of fluctuation in temperature will be accounted for by proper characteristic parameters.
6. The volume occupied by the solid particles is neglected; so is the gravity effect.
7. The solid particles, owing to their small size and high thermal conductivity (as compared with those of the gas), are assumed to be at uniform temperatures.

Steady motion of a gas-solid suspension through a duct of variable area involving friction and heat transfer are governed by the following equations:

Continuity equations—consideration of continuity in the gaseous phase only gives for a perfect gas

$$\frac{m}{A} = \frac{u}{v} = \frac{up}{RT} \quad (1)$$

The concentration of the solid particles is given by

$$m_s = m_p \frac{u}{u_p} \quad (2)$$

Equation (2) accounts for the fact that lower velocity of solids than that of gas shows up as an increase in the concentration of solids.

Over-all energy equation—Consideration of heat transfer from the wall to the gas by convection, and to the solids by radiation, gives, for negligible heat transfer due to collisions of solid particles with the wall

$$d\left[cT + \frac{u^2}{2g_c} + m_s c_p T_p + m_s \frac{u^2_p}{2g_c} \right] = \frac{4(N_{Nu})_d k}{D^2_H} \left(\frac{A}{m}\right) \left[T_w - \left(T + \frac{u^2}{2g_c}\right) \right] dx + \frac{6\,\sigma\,\epsilon\, m_s}{\rho_p d} (T^4_w - T^4_p) \frac{dx}{u_p} \quad (3)$$

for a unit mass of the gas.

The left-hand side of Equation (3) gives the change of total enthalpy of the stream; the right-hand side gives, in the first term, the convection from the wall to the gas and in the second term the radiation from the wall to the gas and in the second term the radiation from the wall to the solids.

Momentum of solid particles—The solid particles are transported by the viscous drag exerted by the gaseous stream. The rate of change of momentum can be expressed as (5)

$$u_p \frac{du_p}{dx} = F\,(u - u_p) \quad (4)$$

where the time lag of fluid-solid motion (sec.$^{-1}$) $F = \frac{3}{4} \frac{C_D}{d} \frac{\rho}{\rho_p} |u - u_p|$, and the drag coefficient is a function of particle Reynolds number $(N_{Re})_p = \frac{d|u - u_p|\rho}{\mu}$. For very small particles Stokes' approximation may be used:

$$F = \frac{18\mu}{\rho_p d^2} \quad (5)$$

It should be noted that μ and ρ are variables.

Over-all momentum equation—Since the solid particles are driven by the gaseous phase, momentum balance gives (6)

$$\frac{dp}{p} + \frac{du^2}{2g_c RT} + \frac{F(u - u_p)m_p}{g_c RT} dx + \frac{2u^2 f}{g_c RT D_H} dx = o \quad (6)$$

The wall friction factor may be generalized to account for the impact of solids with the wall.

Heat transfer from particle to stream—The solid particle receives heat from the wall by radiation, while it transfers heat to the gas by convection:

$$-u_p \frac{dT_p}{dx} = 6\,(N_{Nu})_p \left(\frac{k}{c_p \rho_p d^2}\right) (T_p - T) - \frac{6\,\epsilon\,\sigma}{d c_p \rho_p} (T_w^4 - T_p^4) \quad (7)$$

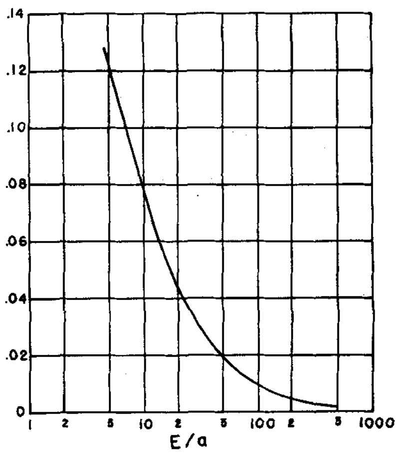

Fig. 1. Velocity lag of solid particles at the exit of a nozzle.

The particle Nusselt number can be approximated by (7):

$$(N_{Nu})_p = \frac{h_p d}{k} = 2 + 0.6\,(N_{Pr})^{1/3}\,(N_{Re})^{1/2} \quad (8)$$

$(N_{Nu})_p$ here may be modified empirically to include turbulent diffusion. For small $(N_{Re})_p$, $(N_{Nu})_p = 2$ the heat transfer will then be simply due to thermal diffusion. In the slip flow and free molecule flow range (3) can be applied.

The above Equations (1), (2), (3), (4), (6) and (7), together with initial conditions, duct shape, and wall temperature descriptions, can be solved simultaneously to determine T, T_p, p, u, and u_p. Instead of the usual technique of nondimensionalizing in terms of Mach number a different method is used because sonic velocity of a two-phase fluid is affected by the momentum and heat transfer between the two phases.* The above equations are nondimensionalized according to reference (say, inlet) temperature, pressure, and characteristic length (length of flow passage):

$$A^* = \frac{A}{m}\frac{U_o}{v_o} = \frac{AU_o p_o}{mRT_o},\ x^* = \frac{x}{L},\ p^* = \frac{p}{p_o}$$

$$u^* = \frac{u}{U_o},\ u^*_p = \frac{u_p}{U_o},\ U_o = \sqrt{2g_c c T_o}$$

$$T^* = \frac{T}{T_o},\ T^*_p = \frac{T_p}{T_o},\ T^* = \frac{T_w}{T_o} \quad (9)$$

Substitution into the gasic equations gives

$$A^* u^* p^* = T^* \quad (10)$$

$$m_s = m_p u^*/u^*_p \quad (11)$$

$$d\left[T^* + u^{*2} + \frac{m_s c_p}{c} T^*_p + m_s u^{*2}_p \right] = B_w A^* \left[T^*_w - T^* - u^{*2}\right] dx^* + \frac{c_p}{c} D m_p \left(T^{*4}_w - T^{*4}_p\right) \frac{dx^*}{u^*_p} \quad (12)$$

$$u^*_p \frac{du^*_p}{dx^*} = E\,(u^* - u^*_p) \quad (13)$$

$$\frac{dp^*}{p^*} + \frac{c}{R}\frac{du^{*2}}{T^*} + \frac{m_s c}{R}\frac{du^{*2}_p}{T^*} + \frac{4c}{R}\frac{fL}{D_H}\frac{u^{*2}dx^*}{T^*} = 0 \quad (14)$$

and

$$-u^*_p \frac{dT^*_p}{dx^*} = B\,(T^*_p - T^*) - D\,(T^{*4}_w - T^{*4}_p) \quad (15)$$

where the parameters

$$B_w = \frac{4(N_{Nu})_d kL}{D^2_H\, c}\,\frac{RT_o}{U_o p_o}$$

relate the heat transfer by convection from the wall to the gas

$$B = \frac{6(N_{Nu})_p kL}{c_p \rho_p d^2 U_o}$$

relates the heat transfer by convection from the solid particles to the gas

$$D = \frac{6\,\epsilon\,\sigma\,T_o\,L}{d\,\rho_p\,c_p\,U_o}$$

relates the heat transfer by radiation from the wall to the solid particles

$$E = \frac{FL}{U_o}$$

relates the momentum transfer from the gas to the solid particles, and

* The formulation in (10) based on the Mach number of the gaseous phase alone is fundamentally incorrect as can be seen in Figure 6.

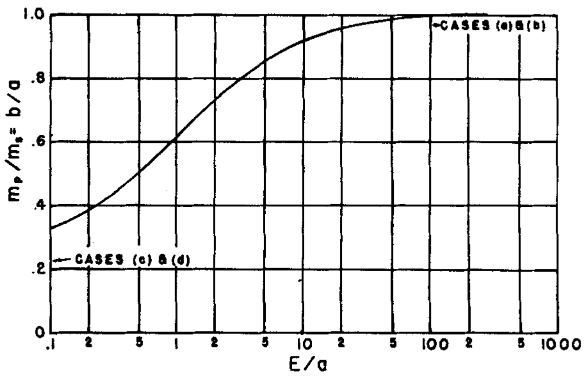

Fig. 2. Change in solid particle concentration due to lag in velocity.

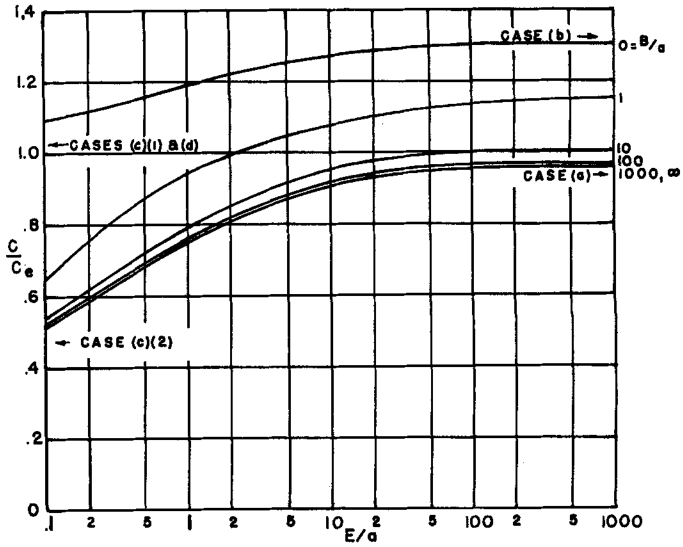

Fig. 3. Specific heat of a two-phase fluid as affected by momentum and heat transfer (0.3 lb. magnesium oxide/lb air).

$$E_w = \frac{fL}{D_H}$$

relates the friction at the wall and the momentum of the fluid. The above are first order simultaneous equations, but their nonlinearity renders a general solution difficult for problems with given $A^*(x^*)$.

In the case of abiabatic ($B_w = o$), flow through a nozzle with negligible wall friction ($f = o$), Equation (12) becomes for very small diameters of solid particles $\epsilon \to 0$ because the particles of diameters smaller than the wave length, where the energy of radiation is a maximum, are transparent to the radiation; thus $D \to 0$:

$$T^* + u^{*2} + \frac{m_s c_p}{c} T^*_p + m_s u^{*2}_p = \frac{c + m_s c_p}{c} \quad (16)$$

Equation (14) becomes

$$\frac{dp^*}{p^*} = -\frac{c}{R}\left[\frac{du^{*2}}{T^*} + m_s \frac{du^{*2}_p}{T^*}\right] \quad (17)$$

and Equation (15) becomes

$$u_p \frac{dT^*_p}{dx^*} = B\,(T^* - T^*_p) \quad (18)$$

together with Equations (10), (11), and (13). The initial conditions being at $x^* = o$

$$u^* = u^*_p = o$$

$$T^* = T^*_p = p^* = 1$$

Numerical solution of Equations (10), (11), (16), (17), and (18) can be made, and so can the solution for the general case involving Equations (10) to (15). However analytical or semi-analytical solution facilitates the basic understanding.

Equations (13), (16), and (18) apply to steady adiabatic flow and solution in closed form and are of definite interest. [For the case involving heat transfer from the wall Equation (16) may be modified in finite difference form to account for a very small step of the integration or replaced by Equation (12).] Elimination of T^* and x^* gives

$$(u^* - u^*_p)\frac{dT_p}{du^*_p} = \frac{B}{E}\left[\frac{c + m_p c_p}{c} - u^{*2} - \frac{m_p c_p}{c}\frac{u^*}{u^*_p}T^*_p - m_p u^* u^*_p - T^*_p\right] \quad (19)$$

which indicates T^* (u^*, u^*_p) and T^* (u^*, u^*_p) as an alternate to x^* as independent variable. Solution can be carried out in terms of u^* and u^*_p. Subsequent integration of Equation (17) and final matching with Equation (10) gives the solution for any flow-area distribution. It is of interest to note that for small solid particles

$$\frac{B}{E} = \frac{2}{3}\frac{c\rho}{c_p \rho_p}/(N_{Pr}) \quad (20)$$

independent of the particle diameter. Since c and c_p are of similar order of magnitude, (N_{Pr}) is of order 1 for a gas, and usually $\rho_p > \rho$, the quantity B/E is usually small. However there are many possible combinations.

A corresponding relation can be written for the case in which there is heat transfer from the wall, and integration can be carried out step-by-step.

SIMPLE CASES

Nozzle Flow

First, simple cases with constant acceleration are considered to obtain some insight into the problem. Flow through a de Laval nozzle is taken for the major example. Reference 2 to 4 considered these four limiting cases:

(a) $u = u_p$, $T = T_p$

(b) $u = u_p$, $T_p = T_o$

(c) $u_p = o$, $T = T_p$

(d) $u_p = o$, $T_p = T_o$

Intermediate situations between (a) and (b) were discussed in reference 3.

In general $u > u_p$ owing to lag of solid particles in an accelerating stream. The solution of equations for given area variations with respect to x is quite complicated owing to nonlinearity of Equation (13). However a simple solution is made first with uniform acceleration assumed. A^* (x^*) is then calculated subsequently. When one takes

$$u^* = ax^* \quad (21)$$

Equation (13) is readily solved to give

$$u^*_p = \frac{1}{2}\left[\sqrt{E^2 + 4Ea} - E\right] x^* = bx^* \quad (22)$$

and

$$b = a\left[1 - \left(\frac{a}{E}\right) + 2\left(\frac{a}{E}\right)^2 - 5\left(\frac{a}{E}\right)^3 + \ldots\ldots\right] \quad (23)$$

($b \to o$ as $E \to o$). The deviation $(u^* - u^*_p)/a$ is given in Figure 1. For given m_p pound of solid per pound of gas initially the concentration variation due to lag in u^*_p is given by Equation (11)

$$\frac{m_s}{m_p} = \frac{u^*}{u^*_p} = \frac{a}{b} \quad (24)$$

In other words the concentration of solid particles increases with greater lag in u^*_p (Figure 2). The gas temperature from the solutions of Equations (16) and (18) is given by

$$T^* = 1 - \frac{c}{c_e} a^2 x^{*2} \quad (25)$$

and

$$\frac{c_e}{c} = \frac{\frac{B}{a}\left(1 + m_s \frac{c_p}{c}\right) + 2\left(\frac{b}{a}\right)}{\left(1 + m_s \frac{b^2}{a^2}\right)\left(\frac{B}{a} + 2\frac{b}{a}\right)} =$$

$$\frac{\frac{B}{a}\left(\frac{b}{a} + m_p \frac{c_p}{c}\right) + 2\left(\frac{b}{a}\right)^2}{\left(1 + m_p\frac{b}{a}\right)\frac{b}{a}\left(\frac{B}{a}+2\frac{b}{a}\right)} \qquad (26)$$

when converted to the throughput m_p. c_e is a measure of amount of heat to be added to a gas-solid suspension in order to raise the temperature of the gas. It is interesting to note that specific heat, normally a thermodynamic property, is transport process dependent in the case of the gas-solid suspension because of nonequilibrium between the two phases. The solid is at different temperatures of the gas. The limiting cases are (corresponding to cases outlined in the above):

(a) $E \to \infty, \frac{b}{a} \to 1,$

$$B \to \infty, c_e = \frac{c + m_p c_p}{1 + m_p}$$

like a heavy gas-light gas mixture:

(b) $E \to \infty, \frac{b}{a} \to 1,$

$$B \to o, c_e = \frac{c}{1 + m_p}$$

with the solid particles remaining at constant temperature:

(c) $E \to o, \frac{b}{a} \to o, B \to \infty$, gives, for

(1) $m_p = o,\ c_e = c$

with solid particles left behind:

(2) $m_p \neq o,\ c_e \to \infty$

with solid particles trickling out at very low velocity and heating the gas phase. The expansion is at constant temperature, with solid particles heating the gas continuously:

(d) $E \to o, \frac{b}{a} \to o, B \to o, c_e = c$

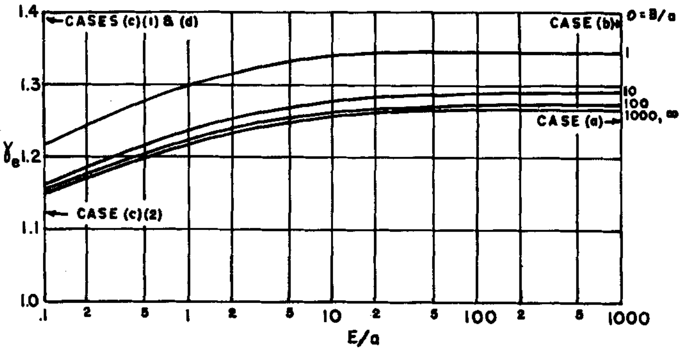

Fig. 5. Apparent ratio of specific heats of a two-phase fluid (0.3 lb. magnesium oxide/lb. air).

in the case of a clean gas.

Figure 3 shows the intermediate cases for a mixture of 0.3 magnesium oxide/lb. of air with air. The temperature difference between the gas and the solid particles is given by

$$\frac{T^*_p - T^*}{\frac{c}{c_e} a^2 x^{*2}} = \frac{2\frac{b}{a}}{\frac{B}{a} + 2\frac{b}{a}} \qquad (27)$$

Figure 4 shows the intermediate and limiting cases of temperature difference between the solid and the gas for the same example of Figure 3.

In general for E and B both greater than 1,000 the approximation of $u = u_p$, $T = T_p$ should be quite accurate.

The apparent ratio of specific heats γ_e of the solid-gas suspension is affected by the condition of momentum and heat transfer:

$$\frac{\gamma_e}{\gamma_e - 1} = \frac{c}{R}$$

$$\frac{\frac{B}{a}\left[\frac{m_p c_p}{c} + \frac{b}{a}\right] + 2\left(\frac{b}{a}\right)^2}{\left(\frac{B}{a} + 2\frac{b}{a}\right)\frac{b}{a}} = \frac{c_e}{R_e} \qquad (28)$$

This relation is shown in Figure 5. In general without heat transfer ($B = o$) the ratio of specific heat is not affected by the solid phase at all. When the drag is very small, γ_e approaches 1 owing to very large concentration of solid particles (case $u_p \sim o$, $T \sim T_p \sim T_o$). However if the solids are not transported at all, the condition reverts to the case of a clean gas system.

The critical velocity of the mixture can be shown to be

$$u^2_c = g\,\gamma_e\,R_e\,T_c \qquad (29)$$

where

$$T_c = \frac{2}{\gamma_e + 1} T_o \qquad (30)$$

and the apparent gas constant

$$R_e = \frac{R}{\left(1 + m_p \frac{b}{a}\right)} \qquad (31)$$

The velocity of sound in a two-phase medium as affected by the transport processes is shown in Figure 6, giving the ratio

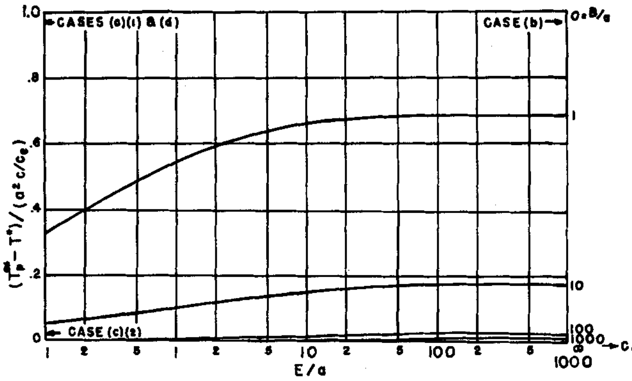

Fig. 4. Temperature difference between the solid and the gaseous phases as affected by momentum and heat transfer (0.3 lb. magnesium oxide/lb. air).

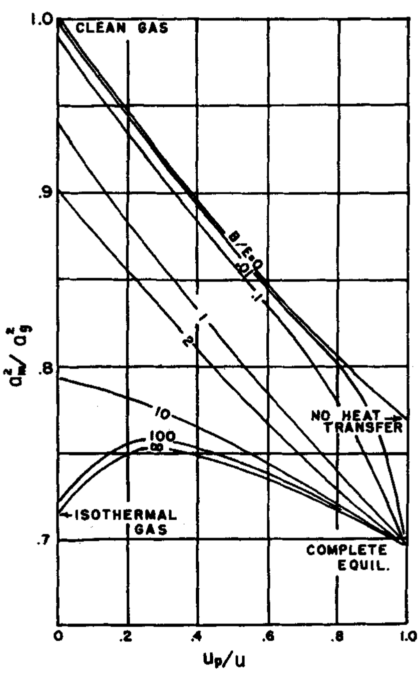

Fig. 6. Velocity of sound through a two-phase fluid (0.3 lb. magnesium oxide/lb. air).

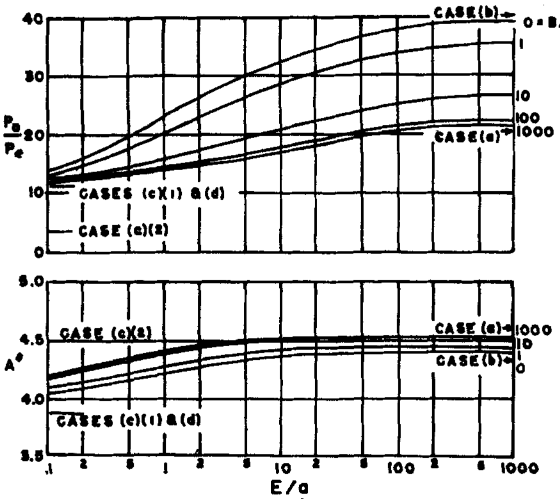

Fig 7. Pressure ratio and flow area requirements of a two-phase fluid (0.3 lb. magnesium oxide/lb. air).

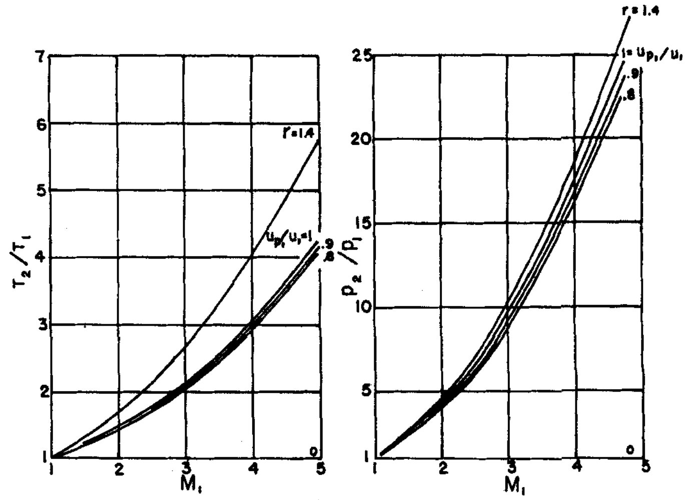

Fig. 9. Temperature and pressure ratios of normal shock of a two-phase fluid (0.3 lb. magnesium oxide/lb. air).

$$\frac{a^2_m}{a^2_g} = \frac{\gamma_e R_e}{\gamma R} = \frac{\dfrac{m_p c_p + c}{c} - \left(1 - 2\dfrac{E}{B}\right)\left(1 - \dfrac{b}{a}\right)}{\left(1 + m_p \dfrac{b}{a}\right)\left[\gamma \dfrac{m_p c_p + c}{c} - (\gamma - 1) - \left(1 - 2\dfrac{E}{B}\right)\left(1 - \dfrac{b}{a}\right)\right]}$$

due to dispersion of sound by the solid particles. This is proven also from wave consideration (*11*).

The throat area of the nozzle (based on 1 lb. of gas) is given by Figure 7:

$$A^*_c = \frac{2c/c_e}{\gamma_e - 1}\left[\left(\frac{\gamma_e + 1}{2}\right)^{\frac{\gamma_e+1}{\gamma_e-1}}\right]^{1/2} \qquad (32)$$

Figure 7 shows that at large B and E heating by the solids increases the volume of the gas; at low values of E the gas velocity tends to the case of a clean gas.

The over-all pressure ratio is given by

$$\frac{p_e}{p_o} = \left[1 - \frac{c}{c_e} a^2\right]^{\frac{\gamma_e}{\gamma_e-1}} \qquad (33)$$

The value of p_o/p_e for $a^2 = 0.5$ is also given in Figure 7, which shows that at low values of B the gas has to be expanded through a much greater pressure ratio to drive the solids. At larger values of B the solids contribute to the energy of expansion by heating the gas.

Worth noting is the case of large B and small E [case (c) (2)]. If this occurs, the situation corresponds to constant temperature expansion of the gas with heating by solids trickling out with the gas. Thus the cross-sectional area at the sonic condition is given by

$$A^* = \sqrt{\frac{2}{\gamma - 1}}\, e^{\gamma/2} \qquad (34)$$

and

$$\frac{p_o}{p_e} = e^{a\sqrt{\gamma/(\gamma-1)}} \qquad (35)$$

as shown in Figure 7. If this condition can be approximated in practice, one should be able to obtain high thrust with low pressure ratio [usually however $B/E < 1$, Equation (20)].

The specific impulse is given by

$$I = \frac{a\sqrt{2g_c c T_o}}{g_c(1 + m_p)}\left[1 + m_s \frac{b}{a}\right] = \frac{a\sqrt{2g_c c T_o}}{g_c} \qquad (36)$$

Hence for the same exit velocity from the nozzle the specific impulse of a gas-solid system is always the same as that of a pure gas of similar composition as the gas phase, regardless of the exit velocity and concentration of the solid phase. The inlet pressure to the nozzle usually has to be higher when solid particles are present. Since B and E are primarily dependent on the length of the passage and E actually

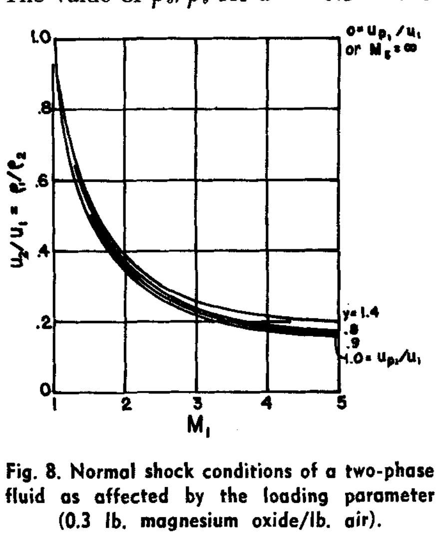

Fig. 8. Normal shock conditions of a two-phase fluid as affected by the loading parameter (0.3 lb. magnesium oxide/lb. air).

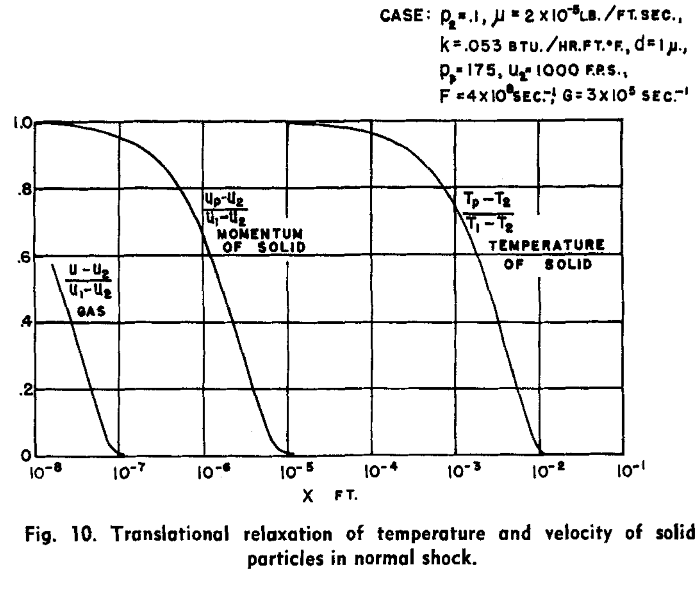

Fig. 10. Translational relaxation of temperature and velocity of solid particles in normal shock.

decreases toward the exit, it can be said that even without wall friction the final state of expansion is affected by the shape and size of the nozzle. In the case of the experiment in reference 8 $B \sim 50$, $E \sim 500$ ($T_o \sim 2{,}990°$R., $d = 0.2\ \mu$, $L = 5.66$ in.); hence considerable difference in temperature between the solid and the gas can be expected. In accordance with the reported gas and solid composition, the present method gives $I = 152$ sec. which is actually lower than measured (157 sec.). The method used in reference 8 gives however a theoretical value of 203 sec. The fact that the present method gives a lower figure may be attributed to change in composition of the fluid during expansion, or even nonequilibrium in the composition, as well as nonuniform acceleration.

Normal Shock

Another interesting phenomenon is the passage of a two-phase stream through a normal shock process. Equations (1), (3), and (6) become

$$u_1\rho_1 = u_2\rho_2 \qquad (37)$$

$$cT_1 + \frac{u_1^2}{2g_c} + m_{s1}c_pT_{p1} + m_{s1}\frac{u_{p1}^2}{2g_c} = cT_2 + \frac{u_2^2}{2g_c} + m_{s2}c_pT_{p2} + \frac{m_{s2}u_{p2}^2}{2g_c} \qquad (38)$$

and

$$p_1 + \rho_1 u_1^2 + m_{s1}\rho_1 u_{p1}^2 = p_2 + \rho_2 u_2^2 + m_{s2}\rho_2 u_{p2}^2 \qquad (39)$$

for the conditions before (state 1) and after (state 2) the shock. Consider the case in which

$$T_p = T,\ u_{p1} \neq u_1,\ u_{p2} = u_2 \qquad (40)$$

that is the particles and gas enter the shock at different velocities but arrive at the same velocity after the shock:

$$\frac{p_2}{p_1} = \frac{\left(\frac{\gamma_e + 1}{\gamma_e - 1}\right) + \left(\frac{\beta_1}{2} + \beta_2\right) - (1 - \beta_1)\frac{\rho_1}{\rho_2}}{\left(\frac{\gamma_e + 1}{\gamma_e - 1}\right)\frac{\rho_1}{\rho_2} - 1} \qquad (41)$$

where

$$\beta_1 = \frac{m_p\rho_1u_1^2}{p_1}\left(1 - \frac{u_{p1}}{u_1}\right) = \frac{m_p\gamma_e}{1 + m_p}(N_M)_1^2\left(1 - \frac{u_p}{u_1}\right)$$

(contribution of momentum transfer)

$$\beta_2 = \frac{m_pc_p}{R}\left(\frac{u_1}{u_{p1}} - 1\right)$$

(contribution of heat capacity) and γ_e and N_{M_1} are defined for the equilibrium situation:

$$\frac{\gamma_e}{\gamma_e - 1} = \frac{c + m_pc_p}{R};\ R_e = \frac{R}{1 + m_p}$$

The velocity ratio u_2/u_1 is given by

$$(N_M)_1^2 = \frac{\frac{2\gamma_e}{\gamma_e - 1}\left(1 - \frac{u_2}{u_1}\right) - \beta_1\left(1 - \frac{u_2}{u_1}\right) + \frac{3\beta_1}{2} + \beta_2}{\left[\gamma_e\frac{\gamma_e + 1}{\gamma_e - 1}\frac{u_2}{u_1} - 1\right]\left[1 - \frac{m_p\left(1 - \frac{u_{p1}}{u_1}\right)}{1 + m_p} - \frac{u_2}{u_1}\right]} \qquad (42)$$

For $u_{p1} = u_1$ the result is identical to the case of shock in a gas of γ_e. For $m_p = u_{p1} = o$, one has simply the case of shock in a clean gas of γ. $(N_M)_1$ is the initial Mach number of the mixture.

Again with 0.3 lb. magnesium oxide/lb. of air for an example the velocity or density ratio (u_2/u_1 or ρ_1/ρ_2) through a shock is shown in Figure 8 for various values of u_{p1}/u_1 as compared with air alone. The particles contribute to inertia against a strong shock. For very large m_p or very small u_p (and therefore large m_s) no shock is possible.

Figure 9 shows the pressure ratio (p_2/p_1) and temperature ratio (T_2/T_1) of shock at various initial Mach numbers. It can be seen that the pressure recovery in normal shock of a two-phase fluid is much lower than the clean gas, especially where there is large velocity lag of solid particles.

Owing to heat capacity and mass inertia, there arises a shock thickness due to the presence of solid particles, in terms of the coordinate in the direction of motion

$$\frac{u_p - u_2}{u_1 - u_2} \cong e^{-\frac{Fx}{u_2}} \qquad (43)$$

$$\frac{T_p - T_2}{T_1 - T_2} \cong e^{-\frac{Gx}{u_2}} \qquad (44)$$

where $G = 6\,(N_{Nu})_p\,\frac{k}{c_p\rho_pd^2}$, on the assumption that the viscous shock thickness of the gas is much smaller than the thickness due to inertia of particles. Since $\frac{F}{G} = \frac{3}{2}\,\frac{c_p\rho_p}{c_p}\,(N_{Pr})$, it is expected that the thermal shock thickness is greater than the momentum shock thickness. The orders of magnitude are shown in Figure 10. The shock thickness in the gas is in this case of the order of 10^{-7} ft. ($\delta \sim \mu(u_1 - u_2)/\rho$). This is an interesting case of translational relaxation.

SOLUTION FOR ARBITRARY AREA DISTRIBUTION

When a nozzle with an arbitrary area distribution is used to handle a two-phase stream, the velocities are affected by the momentum and heat transfer in a more complicated manner as far as calculation is concerned. Equation (19) can be integrated in the following parametric form:

$$T^*_p = \left(\frac{c}{c + m_p\,c_p}\right)\left\{\left[\left(\frac{c + m_p\,c_p}{c}\right) - u^{*2}\right]\left[1 - \frac{r}{s - 1}\left(\frac{u^*}{u^*_p} - 1\right) + \frac{r(r - 1)}{(s - 1)(s - 2)}\left(\frac{u^*}{u^*_p} - 1\right)^2 - \ldots\ldots\right] - m_p\,u^{*2}\left(\frac{u^*_p}{u^*}\right)\left[1 - \frac{r + 1}{s - 1}\left(\frac{u^*}{u^*_p} - 1\right) + \frac{(r + 1)r}{(s - 1)(s - 2)}\left(\frac{u^*}{u^*_p} - 1\right)^2 - \ldots\ldots\ldots\right] + (1 + m_p)\,u^{*2}\left(\frac{u^*}{u^*_p} - 1\right)^s\left(\frac{u^*_p}{u^*}\right)^{s - r}\right\} \qquad (45)$$

where

$$r = \frac{B}{E}\frac{c_pm_p}{c},\ s = \frac{B}{E}\left(\frac{c + m_p\,c_p}{c}\right),\ s - r = \frac{B}{E}$$

Equation (45) satisfies the initial condition $u^* = o$, $T^*_p = 1$, and the limiting conditions

$$\frac{B}{E} = o,\ T^*_p = 1$$

and

$$\frac{B}{E} = \infty,\ T^*_p = T^* =$$

$$\frac{\left(\frac{c + m_p c_p}{c}\right) - u^{*2} - m_p u^* u^*_p}{\left(\frac{c + m_p c_p}{c}\right) + \frac{m_p c_p}{c}\left(\frac{u^*}{u^*_p} - 1\right)}$$

$$T^* = \frac{c + m_p c_p}{c} - u^{*2}\left(1 + m\frac{u^*_p}{u^*}\right) - \frac{m_p c_p}{c}\frac{u^*}{u^*_p} T_p \quad (46)$$

Equation (17) enables determination of p^* numerically, and hence $A^*\left(u^*, \frac{u^*_p}{u^*}\right)$ for given B/E. Matching with $A^*(x^*)$ while satisfying Equation (13) one gets $u^*(x^*)$ and other properties or functions of x^*.

To illustrate this method two differently shaped nozzles, both having area distribution for isentropic expansion of air ($\gamma = 1.40$) at a pressure ratio of 200, were taken for the example. Nozzle I is for uniform acceleration of air:

$$A^{*-1} = K_n x^* \left[1 - K_n^2 x^{*2}\right]^{\frac{1}{\gamma - 1}} \quad (47)$$

and

$$K_n^2 = 1 - \left(\frac{p_e}{p_o}\right)^{\frac{\gamma - 1}{\gamma}}$$

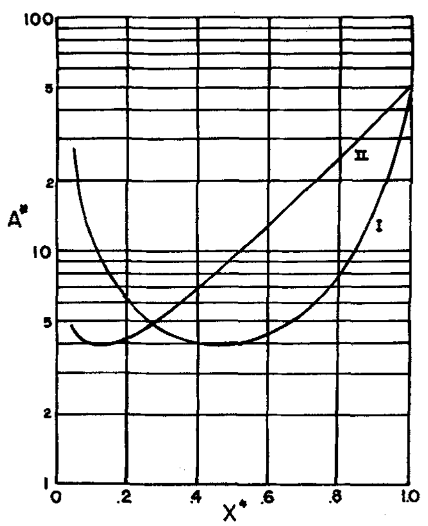

Fig. 11. Area distribution of nozzles in the numerical calculation ($\gamma = 1.40$, $p_o/p_e = 200$).

Nozzle II is for expansion of air at uniform pressure ratio:

$$A^{*-1} = e^{K_m x^*/(\gamma-1)} \left[1 - e^{K_m x^*}\right]^{1/2} \quad (48)$$

and

$$K_m = \frac{\gamma - 1}{\gamma} \ln (p_o/p_e)$$

The area variations of nozzles I and II are shown in Figure 11. A suspension with 0.3 lb. magnesium oxide/lb. air is again taken for the example. The results of a few cases calculated by numerical solutions are summarized in the following table. (The last row gives the weight flow rate in pound per second for a nozzle designed for clean air at the rate of 1 lb./sec.)

Flow through Nozzles designed for air with $p_o/p_e = 200$, $u_e^* = 0.8824$

B/E	0		0		0.1		0.1	
E	10		50		10		50	
Nozzle form	I	II	I	II	I	II	I	II
u_e^*	0.775	0.776	0.780	0.780	0.798	0.803	0.811	0.817
p_o/p_e	233		244		210	221	130	136
$\frac{m(1 + m_p)}{m_a}$	1.188	1.256	1.112	1.162	1.1152	1.208	1.111	1.133

The above table shows that the shape of a nozzle affects its performance when handling a two-phase suspension, even if wall friction is neglected. At high heat transfer rate between the solid particles and the stream a nozzle tends to be under expanded, while at low values of B over expansion and shock may result. The through flow is affected by the rate of expansion; higher initial rate calls for smaller throat area.

It should be noted that only simple cases of constant B and E are illustrated in the above. In general both values decrease toward the exit of a nozzle. The case with high acceleration initially would involve greater value of B owing to convection heat transfer.

CONCLUDING REMARKS

A method has been developed such that all one-dimensional gas dynamic problems of a two-phase suspension can be computed. For duct flow with heat transfer step by step integration can be used.

From consideration of simple cases one has seen that in flow through nozzles the results of reference 2, 3, 4 are confirmed as limiting cases. It has been generalized that it is not just a matter of small particles and dense gas that will produce small momentum and thermal lag. It calls for large heat transfer parameter

$$B = 6(N_{Nu})_p \frac{k}{c_p \rho_p} \frac{L}{d^2 U_o}$$

and large drag parameter

$$E = \frac{3}{4} \frac{C_D}{d} \frac{\rho_p}{\rho} \frac{|u - u_p|}{U_o} L \sim \frac{18 \mu L}{\rho_p d^2 U_o}$$

The ratio

$$\frac{B}{E} = \frac{2}{3} \frac{c\rho}{c_p \rho_p} / N_{Pr}$$

is usually small; hence thermal lag is usually more significant than momentum lag, although the latter might be quite large at the exit of a nozzle. For particle size smaller than the mean free path Λ (3, 9)

$$\frac{B}{E} = \frac{1}{\sqrt{2\pi}} \frac{c''_v}{c'_p} \alpha' \left[f'\left(\frac{d}{\Lambda} - 1\right) + 2 \right] \quad (49)$$

The drag parameter

$$E = \frac{6\sqrt{2\pi}\, pL}{d\rho_p \sqrt{RT}\, U_o} \frac{\Lambda}{d} \left[f'\left(1 - \frac{\Lambda}{d}\right) + 2\frac{\Lambda}{d} \right] \quad (50)$$

may have a very small value.

The velocity of sound through a gas-solid suspension is another point of interest. This velocity is affected by the heat and momentum transfer between the two phases. For large drag and small heat transfer the solid contributes only to the molecular weight. If the heat transfer rate is also large, the situation of a heavy and light gas mixture arises. If the drag is very small, the solid particles do not contribute to the velocity of sound. Therefore any gas dynamic analysis referring to the velocity of sound of the gas phase alone is incorrect.

Turbulence, when present, tends to increase the value of B. It also increases the value of E in Equation (14). In general B, B_w, D, E, and E_w are variables.

In the numerical examples convection from solid particles was neglected. This contribution moves the curves at the end of low values of E toward the direction of greater values of B.

These considerations indicate that even without wall friction the shape of the nozzle affects its thrust. For a nozzle operating with very low exit pressure the expansion should be as complete as possible because the particles do not contribute to pressure thrust. The presence of solid particles reduces the tendency toward shock discontinuity.

While it is desirable to have nearly complete expansion in a nozzle, over expansion and shock is far less desirable. The solid particles which are slowed down do not contribute to pressure thrust. The pressure recovery is lower than that of clean gas, especially when there is large lag in particle velocity.

All these suggest more stringent requirements of an efficient nozzle for a rocket system containing a solid phase.

The case involving change of phase and composition of fluid during expansion through a nozzle would be naturally the next step of exploration.

In the thermodynamic sense the gas-solid suspension is an interesting system, of which the usual thermodynamic properties, such as specific heat, velocity of sound, etc., are transport process dependent because of the inherent nonequilibrium.

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NOTATION

A = flow area, sq. ft.

A_c = critical or throat flow area, sq. ft.

a = linear acceleration of the gas, dimensionless, $u^* = ax^*$

a_m = velocity of sound in the mixture, ft./sec.

a_g = velocity of sound in the clear gas, ft./sec.

B = convection heat transfer parameter between solid particles and gas, dimensionless,

$$B = \frac{6(N_{Nu})_p kL}{c_p \rho_p d^2 U_o}$$

B_w = convection heat transfer parameter between wall and fluid, dimensionless,

$$B_w = \frac{4(N_{Nu})_d kL}{D_H^2 c} \frac{RT_o}{U_o p_o}$$

b = linear acceleration of solid particles, dimensionless, $u^*_p = bx^*$

c = specific heat at constant pressure of gas, B.t.u./lb. °R.

C_D = drag coefficient of relative motion between solid particles and fluid, dimensionless

c_p = specific heat of solid particles, B.t.u./lb. °R.

c_e = apparent specific heat at constant pressure of two-phase fluid, B.t.u./lb. °R.

c''_v = molal translational specific heat at constant volume of gas, B.t.u./lb. mole °R.

c'_p = molal specific heat of solid particles, B.t.u./lb. mole °R.

D = radiation heat transfer parameter, dimensionless,

$$D = \frac{6 \epsilon \sigma T_o^3 L}{d \rho_p c_p U_o}$$

D_H = hydraulic diameter of flow passage, ft.

d = diameter of solid particles or characteristic diameter of nonspherical shape, ft.

E = drag parameter between solid particles and gas, dimensionless, $E = FL/U_o$

E_w = friction parameter between wall and fluid, dimensionless, $E_w = fL/D_H$

F = time lag of fluid-solid motion, (sec.)$^{-1}$

f = friction factor of flow passage, dimensionless

f' = velocity slip coefficient, dimensionless

g_c = gravitational constant relating force and mass, lb.-mass ft./lb.-force sec.^{2}

G = thermal lag, sec.$^{-1}$,

$$G = \frac{6(N_{Nu})_p k}{c_p \rho_p d^2}$$

h_p = convection heat transfer coefficient between solid particles and gas, B.t.u./sq. ft. °R. sec.

k = thermal conductivity of gas, B.t.u./ft. sec. °F.

L = length of flow passage, ft.

m = weight flow rate of gas, lb./sec.

m_p = pound of solid particles per pound of gas of inlet mixture, dimensionless

m_s = pound of solid particles per pound of gas at any location, dimensionless

N_M = initial Mach number of the mixture, dimensionless

$(N_{Nu})_d$ = Nusselt number of flow passage, dimensionless

$(N_{Nu})_p$ = Nusselt number of convection between solid particles and gas, dimensionless

N_{Pr} = Prandtl number of gas, dimensionless

$(N_{Re})_p$ = Reynolds number of relative motion between solid particles and gas, dimensionless

p = static pressure, lb./sq. ft. abs.

p_o = initial static pressure, lb./sq. ft. abs.

p^*_o = p_e/p_o, over-all pressure ratio, dimensionless

R = gas constant of the gas, ft. lb./lb. °R. or B.t.u./lb. °R.

R_e = apparent gas constant of a two-phase fluid, ft. lb./lb. °R. or B.t.u./lb. °R. defined by Equation (31)

T = static temperature of gas, °R.

T_w = wall temperature, °R.

T_p = temperature of solid particles, °R.

T_o = initial static temperature, °R.

T_c = critical temperature, °R.

u = velocity of gas, ft./sec.

u_p = velocity of solid particles, ft./sec.

U_o = characteristic velocity, ft./sec. (= $\sqrt{2g_c cT_o}$ in the solutions)

v = specific volume of gas, cu. ft./lb.

v_o = initial specific volume, cu. ft./lb.

x = space coordinate in the direction of motion, ft.

Greek Letters

ϵ = emissivity based on size and surface of solid particles, dimensionless

ρ_p = density of solid particles, slug/cu. ft.

ρ = density of gas, slug/cu. ft.

Λ = mean free path of gas, ft.

μ = viscosity of gas, lb./ft. sec.

σ = Stefan-Boltzmann constant, 0.172 x 10^{-8} B.t.u./hr. sq. ft. °R.

γ = ratio of specific heats of gas, dimensionless

γ_e = apparent ratio of specific heats of two-phase fluid, dimensionless

Subscripts

o = inlet or reference quantities

1,2 = initial and final conditions

Superscript

(*) = reduced quantities, dimensionless

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